THE EFFECT OF AN ELECTRIC FIELD OF STRENGTH $-1/2aF^2$ ON THE POLARIZABILITY CONSTANT OF THE NORMAL HYDROGEN ATOM*

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ABSTRACT. Using the integral for the upper limit of the energy in its lowest state of a system as [1] $\int\phi^*H\phi d\tau$ and applying the variation method along the z axis gives the energy. With a field strength of $-1/2aF^2$, produces a negative polarizability constant for the normal hydrogen atom. The value obtained is $a=-.58047\times10^{-24}$ cm.[3] which is in close agreement with that obtained by the second-order perturbation theory.

The electric moment of the induced dipole (charge × distance) is aF, where 'a' is called the 'polarizability' of the atom or ion and F is the strength of the field. Using the integral for the upper limit of the energy in its lowest state of a system as

$$^1E=\int\phi^*H\phi d\tau \quad \ldots \quad (1)$$

where H is the complete Hamiltonian operator and is expressed as follows

$$H=T(q,p)+V(q) \text{ and } P=\frac{h}{2\pi i}\cdot\frac{\partial}{\partial q}.$$

For the normal hydrogen atom 1s, with $n=1$, $L=0$, $m=0$ is given by the wave function

$$\psi_{1s}=\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0} \quad \ldots \quad (2)$$

Using the variation method along the z-axis and multiplying eq. (2) by $(1+Az)$,

Letting $$\phi=(1+Az)\psi_{1s}=\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a}(1+Az) \quad \ldots \quad (3)$$

where $z=r\cos\theta$

The Hamiltonian operator will be in this case

$$H=-\frac{h^2}{8\pi^2}\sum\frac{1}{m_i}\nabla_i^2+V \quad\text{and}\quad V=-\frac{e^2}{r}+e.F.z.$$

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Using H on equation 3 gives

$$H\phi = -\frac{h^2}{8\pi^2}\sum \frac{1}{m_i}\nabla_i^2\phi + V\phi$$

$$H\phi = -\frac{h^2}{8\pi^2}\left(\frac{1}{m_1}\nabla_1^2\phi + \frac{1}{m_2}\nabla_2^2\phi\right) + V\phi$$

$$H\phi = -\frac{h^2}{8\pi^2}\left(\frac{1}{\mu}\nabla^2\phi\right) \qquad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$= -\frac{h^2}{8\pi^2}\left[\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2}{\partial\theta^2}\right.$$

$$\left. + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] - \frac{e^2}{r} + eFr\cos\theta.$$

When we substitute $H\phi$ and ϕ in eq. (1) and evaluate over all space, eq. (1) reduces to

$$E = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{e^{-2r/a_0}}{a_0^{3/2}}(1 + Ar\cos\theta)\left[-\frac{h^2}{8\pi^2\mu}\cdot\frac{1}{r^2}\left\{\frac{1}{a_0^2}\left(r^2 + Ar^3\cos\theta\right)\right.\right.$$

$$\left. - \frac{1}{r^2 a_0}(2r + 3r^2\cos\theta) + \frac{2rA\cos\theta}{r^2} - r^2\left(\frac{1}{a_0}\right)A\cos\theta\right\} - \frac{2Ar\sin\theta\cos\theta}{r^2\sin\theta}$$

$$\left. - \frac{e^2}{r}(1 + Ar\cos\theta) + eFr\cos\theta\,(1 + Ar\cos\theta)\right] r^2\sin\theta\, dr d\phi. \quad \cdots \quad (4)$$

With the following integrals and their values,

$$\int_0^{\pi}\sin\theta\, d\theta = 2, \quad \int_0^{\pi}\sin\theta\cos\theta\, d\theta = 0, \quad \int_0^{\pi}\sin\theta\cos^2\theta\, d\theta = 2/3 \text{ and}$$

$$\int_0^{\infty}\sin\theta\cos^3\theta\, d\theta = 0$$ substituted in eq. (4) gives

$$E = \frac{2}{a_0^3}\int_0^{\infty} e^{-2r/a_0}\left[-\frac{h^2}{8\pi^2\mu}\left(\frac{2}{a_0^2} - \frac{4}{a_0 r}\right) - \frac{2e^2}{r} + 2/3\, eFr^2 A\right.$$

$$\left. - \frac{h^2}{8\pi^2\mu}\left(\frac{2/3A^2r^2}{a_0^2} - \frac{2A^2r^2}{a_0} - \frac{8A^2}{3} - \frac{2A^2r}{3a_0}\right) - \frac{2e^2A^2r}{3} + \frac{2eFr^2}{3}\right] dr$$

which reduces to the following, when the integrals of this form

$\int_0^{\infty} e^{-2r/a_0} r^n dr = \left(\frac{a_0}{2}\right)^{n+1} \lfloor n$ are evaluated for values of $n = 1, 2, 3$ and 4.

Their values are respectively $\frac{a_0^3}{4}, \frac{a_0^3}{4}, \frac{3a_0^4}{8}, \frac{3a_0^5}{4}$.

Therefore a reduced form of equation (4) is as follows,

$$E = \left[\frac{h^2}{8\pi^2\mu a_0^2} - \frac{e^2}{a_0} + 2eFAa_0^2 + \frac{h^2A^2}{8\pi^2\mu} - \frac{e^2A^2a_0^2}{2}\right]. \qquad \dots (5)$$

Since it is not convenient to normalize ϕ we can divide eq. (1) by

$$\int \phi^*\phi d\tau$$

and E will retain its validity.

Evaluating $\int \phi^*\phi d\tau$ and integrating over all space,

$$\int \phi^*\phi d\tau = \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{e^{-2r/a_0}}{\pi a_0^3} (1 + rA\cos\theta)^2 r^2 \sin\theta d\theta d\phi dr$$

$$= 2\int_0^\infty \frac{e^{-2r/a_0}}{a_0^3} r^2 dr + \frac{2}{3}\int_0^\infty \frac{e^{-2r/a_0}}{a_0^3} A^2 r^4 dr$$

$$= (1 + a_0^2A^2). \qquad \dots (6)$$

Dividing eq. (5) by eq. (6) and letting $h^2 = 4\pi^2 a_0 \mu e^2$, E has the form

$$E = \frac{1}{1 + a_0^2A^2}\left(-\frac{e^2}{2a_0} + 2eFa_0^2A\right). \qquad \dots (7)$$

Differentiating eq. (7) with respect to A and equating it to zero gives the equation

$$-2a_0^3FA^2 + eA + 2Fa_0 = 0. \qquad \dots (8)$$

Solving for A and omitting $16F^2a_0^4$, because it is small in comparison to the 'e' term, leaves

$$A = \frac{e}{2a_0^3F}.$$

Substituting this value of A in eq. (7) reduces it to the following

$$E = \frac{4a_0^4F^2}{4a_0^4F^2 + e^2}\left(\frac{e^2}{2a_0}\right). \qquad \dots (9)$$

Since E is given equal to $-1/2aF^2$, equation (9) gives

$$a = -4a_0^3,$$

and then taking the radius of the orbit in the normal hydrogen atom, referred to the center of mass as [2] $a_0 = .5282 \times 10^{-8}$ cm., gives $a = -.58947 \times 10^{-24}$ cm.[3]

Therefore, such a field strength produces a negative '*a*' which shows that the induced dipole is in the opposite direction to the field instead of in the same direction. This value of '*a*' obtained here is in close agreement with that obtained by the first- and second-order perturbation theory applied to the Stark effect of the hydrogen atom by Epstein,[3] Wentzel,[4] and Waller.[5]

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